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1991 J. Phys. A: Math. Gen. 24 L179

(http://iopscience.iop.org/0305-4470/24/4/004)

J. Phys. A: Math. Gen. 24 (1991) L179–L183. Printed in the UK

LETTER TO THE EDITOR

q-fermionic operators and quantum exceptional algebras

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Received 29 November 1990

Abstract. Using a q-deformation of the usual Clifford algebra, one constructs a realization of the quantum exceptional algebras E_6, E_7, E_8 and F_4 in terms of q-fermionic operators.

The study of quantum integrable systems has led to new algebraic structures, called quantum groups [1], which are deformations of the enveloping algebra of a Lie algebra, such that one recovers the Lie algebra under consideration in the limit $q \to 1$, where q is the deformation parameter. A particular realization of the infinite series $\mathrm{SU}(n)_q$, $\mathrm{SO}(n)_q$ and $\mathrm{Sp}(2n)_q$ has been obtained by using quantum deformation of the usual Weyl and Clifford algebras [2, 3]. An explicit matrix realization of the $(\mathrm{G}_2)_q$ algebra has already been given in [4]. In this letter, we would like to concentrate on the other exceptional cases and give a realization of the quantum algebras E_6, E_7, E_8 and F_4 in terms of operators satisfying q-deformed Clifford algebras.

Let us recall the definition of a quantum enveloping algebra $\mathrm{U}_q(\mathrm{G})$ associated with a simple Lie algebra G of rank n [1, 3]. Mathematically, the quantum enveloping algebra $\mathrm{U}_q(\mathrm{G})$ is a Hopf algebra with unit 1 and generators E_i^+, E_i^-, H_i $(1 \leqslant i \leqslant n)$ defined through the commutation relations in the Chevalley basis

$$\begin{aligned} &[H_i, H_j] = 0 \\ &[H_i, E_j^+] = a_{ij} E_j^+ \\ &[H_i, E_j^-] = -a_{ij} E_j^- \\ &[E_i^+, E_j^-] = \delta_{ij} \frac{q^{2d_i H_i} - q^{-2d_i H_i}}{q^{2d_i} - q^{-2d_i}} \end{aligned} \tag{1}$$

and the quantum Serre–Chevalley relations (for $i \neq j$)

$$\sum_{0 \leqslant n \leqslant 1-a_{ij}} (-1)^n \begin{bmatrix} 1-a_{ij} \\ n \end{bmatrix}_{q^{2d_i}} (E_i^+)^{1-a_{ij}-n} E_j^+ (E_i^+)^n = 0 \tag{2a}$$

$$\sum_{0 \leqslant n \leqslant 1-a_{ij}} (-1)^n \begin{bmatrix} 1-a_{ij} \\ n \end{bmatrix}_{q^{2d_i}} (E_i^-)^{1-a_{ij}-n} E_j^- (E_i^-)^n = 0 \tag{2b}$$

where (a_{ij}) $(1 \leqslant (i, j) \leqslant n)$ is the Cartan matrix of the Lie algebra G, and d_i are non-zero integers, with greatest common divisor equal to one, such that $d_i a_{ij} = d_j a_{ji}$. Notice that if the Cartan matrix is symmetric, all the d_i's are equal to one.

The q-binomial coefficients $[{m \atop n}]_q$ are defined by

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]_q!}{[m-n]_q![n]_q!} \qquad \text{with} \qquad [m]_q! = [m]_q \ldots [1]_q \qquad \text{and}$$

$$[m]_q = \frac{q_m - q^{-m}}{q-q^{-1}}. \tag{3}$$

One needs also to introduce a comultiplication Δ, a co-unit ε and an antipode S such that

$$\begin{aligned}
&\Delta(H_i) = 1 \otimes H_i + H_i \otimes 1 \\
&\Delta(E_i^{\pm}) = E_i^{\pm} \otimes q^{-H_i} + q^{H_i} \otimes E_i^{\pm} \\
&\varepsilon(E_i^{\pm}) = \varepsilon(H_i) = 0 \qquad \text{and} \qquad \varepsilon(1) = 1 \\
&S(E_i^{\pm}) = -q^{-H_i} E_i^{\pm} q^{H_i} \qquad \text{and} \qquad S(H_i) = -H_i.
\end{aligned} \tag{4}$$

The q-deformation $\mathrm{Cliff}_q(n)$ of the Clifford algebra of dimension n with generators a_i^-, a_i^+ and N_i $(1 \leqslant i \leqslant n)$ is defined by†

$$\begin{aligned}
&a_i^- a_j^+ + q^{2\delta_{ij}} a_j^+ a_i^- = \delta_{ij} q^{2N_i} \\
&a_i^- a_j^+ + q^{-2\delta_{ij}} a_j^+ a_i^- = \delta_{ij} q^{-2N_i} \\
&(a_i^-)^2 = (a_i^+)^2 = 0 \\
&[N_i, a_j^{\pm}] = \pm \delta_{ij} a_j^{\pm}.
\end{aligned} \tag{5}$$

Notice that the q-analogue of the Clifford algebra is invariant in the change $q \leftrightarrow q^{-1}$. This allows us to compute the products $a_i^- a_i^+$ and $a_i^+ a_i^-$ in terms of N_i. One finds

$$a_i^+ a_i^- = \frac{q^{2N_i} - q^{-2N_i}}{q^2 - q^{-2}} \tag{6a}$$

$$a_i^- a_i^+ = -\frac{q^{2N_i-2} - q^{-2N_i+2}}{q^2 - q^{-2}}. \tag{6b}$$

We recall that there exist the following homomorphism between $U_q(D_n)$ and $\mathrm{Cliff}_q(n)$ (see [3]):

$$\begin{aligned}
&E_i^+ \to a_{i+1}^+ a_i^- \qquad E_i^- \to a_i^+ a_{i+1}^- \qquad H_i \to N_{i+1} - N_i \qquad (1 \leqslant i \leqslant n-1) \\
&E_n^+ \to a_n^+ a_{n-1}^+ \qquad E_n^- \to a_{n-1}^- a_n^- \qquad H_n \to N_{n-1} + N_n.
\end{aligned} \tag{7}$$

† There exists another definition of the q-analogue of the Clifford algebra given by (see [3]):

$$\begin{aligned}
&a_i^- a_j^+ + q^{2\delta_{ij}} a_j^+ a_i^- = \delta_{ij} \omega^2 \\
&a_i^- a_j^+ + q^{-2\delta_{ij}} a_j^+ a_i^- = \delta_{ij} \omega^{-2} \\
&(a_i^-)^2 = (a_i^+)^2 = 0 \\
&\omega_i a_j^- \omega_i^{-1} = q^{\delta_{ij}} a_j^- \qquad \text{and} \qquad \omega_i a_j^- \omega_i^{-1} = q^{-\delta_{ij}} a_j^+ \\
&\omega_i \omega_i^{-1} = \omega_i^{-1} \omega_i = 1.
\end{aligned}$$

Although this definition is more general than ours (that one can recover by $\omega_i = e^{-hN_i}$ with $q = e^h$), the definition equation (5) leads to the limit $q \to 1$ (or $h \to 0$) in a more transparent way.

To obtain a realization of the quantum exceptional algebras in terms of q-deformed fermionic operators, let us introduce two sets of fermionic operators, denoted respectively by (a_i^-, a_i^+, N_i) and (b_i^-, b_i^+, M_i) with $i=1$–4. The a-operators and the b-operators are not independent and their mutual action can be computed by

$$\begin{aligned} a_i^+ a_j^+ &= (-1)^{j-i-1} b_k^- b_m^- \\ a_i^+ a_j^- &= b_i^+ b_j^- \\ M_i &= \tfrac{1}{2}\left(N_i - \sum_{k\neq i} N_k \right). \end{aligned} \tag{8}$$

Then, one has

$$M_i - M_j = N_i - N_j \qquad \text{and} \qquad M_i + M_j = -(N_k + N_m). \tag{9}$$

In equations (8) and (9), the set of indices (i, j, k, m) forms a permutation of (1, 2, 3, 4) with $i<j$ and $k<m$ (for example, one has $a_1^+ a_2^+ = b_3^- b_4^-$). Moreover, we require (for $i \neq j$)

$$[b_i^+, a_j^-] = [b_i^+, a_i^+] = 0 \qquad \text{and} \qquad [b_i^+, [b_i^+, a_j^+]] = 0. \tag{10}$$

We also introduce a new copy of a-operators and b-operators labelled by $i=5$–8, which satisfy between them relations analogous to (5) and (8). We do not need to specify other relations between these operators in the following (in particular the operators labelled by $i=1$–4 are independent of the operators labelled by $i=5$–8). The construction is based on the construction of the Lie exceptional algebras in terms of fermionic operators given in [5]; for more detail on the construction in the classical case, we invite the reader to refer to this paper.

The symmetric Cartan matrix of E_8 is given by (see [6])

$$A = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}. \tag{11}$$

A realization of the generators of E_8 in the Chevalley basis in terms of bilinear creation and annihilation operators is given by

$$\begin{array}{llll} E_1^+ \to b_1^+ b_8^+ & E_2^+ \to a_1^+ a_2^+ & E_3^+ \to a_2^+ a_1^- & E_4^+ \to a_3^+ a_2^- \\ E_5^+ \to a_4^+ a_3^- & E_6^+ \to a_5^+ a_4^- & E_7^+ \to a_6^+ a_5^- & E_8^+ \to a_7^+ a_6^- \end{array} \tag{12a}$$

and

$$H_1 \to M_1 + M_8 \qquad H_2 \to N_1 + N_2 \qquad H_i \to N_{i-1} - N_{i-2} \qquad (3 \leqslant i \leqslant 8). \tag{12b}$$

It is easy to verify that all the relations, (1) and (2), defining the universal enveloping algebra $U_q(E_8)$ associated to matrix (11) are satisfied. In order to check equations (1) and (2), it is enough to study them for the first line of the Cartan matrix since the remaining equations correspond to the Cartan matrix of D_7 (see [3]):

$a_{12}=0$:

$$\sum_{n=0}^{1}(-1)^n\begin{bmatrix}1\\n\end{bmatrix}_{q^2}(E_1^+)^{1-n}E_2^+(E_1^+)^n$$
$$=[E_1^+,E_2^+]\approx[b_1^+b_8^+,a_1^+a_2^+]\approx[b_1^+b_8^+,b_3^-b_4^-]=0. \tag{13a}$$

The same arguments hold for a_{1i} with $i=4$–8.

$a_{13}=-1$:

$$\sum_{n=0}^{1}(-1)^n\begin{bmatrix}2\\n\end{bmatrix}_{q^2}(E_1^+)^{1-n}E_3^+(E_1^+)^n$$
$$=[E_1^+,[E_1^+,E_3^+]_{q^{-2}}]_{q^2}\approx[b_1^+b_8^+,[b_1^+b_8^+,a_1^+a_2^+]_{q^{-2}}]_{q^2}$$
$$\approx[b_1^+b_8^+,[b_1^+b_8^+,b_3^-b_4^-]_{q^{-2}}]_{q^2}=0 \tag{13b}$$

$a_{16}=0$:

$$\sum_{n=0}^{1}(-1)^n\begin{bmatrix}1\\n\end{bmatrix}_{q^2}(E_1^+)^{1-n}E_6^+(E_1^+)^n=[E_1^+,E_6^+]\approx[b_1^+b_8^+,a_5^+a_4^-]=0. \tag{13c}$$

The symmetric Cartan matrix of E_7 is given by deleting the last row and the last column of the Cartan matrix (11) of E_8. To obtain a realization of E_7 in terms bilinear creation and annihilation operators, one considers the maximal regular embedding $E_8\supset E_7$. The realization for E_7 is immediately obtained by

$$\begin{array}{llll} E_1^+\to b_1^+b_8^+ & E_2^+\to a_1^+a_2^+ & E_3^+\to a_2^+a_1^- & E_4^+\to a_3^+a_2^- \\ E_5^+\to a_4^+a_3^- & E_6^+\to a_5^+a_4^- & E_7^+\to a_6^+a_5^- & \end{array} \tag{14a}$$

and

$$H_1\to M_1+M_8 \qquad H_2\to N_1+N_2 \qquad H_i\to N_{i-1}-N_{i-2} \qquad (3\leqslant i\leqslant 7). \tag{14b}$$

In the same way, the symmetric Cartan matrix of E_6 is obtained by deleting the last row and the last column of that of E_7. Again, in order to obtain a realization of E_6 in terms bilinear in creation and annihilation operators, one considers the maximal regular embedding $E_7\supset E_6$. Therefore, one gets

$$\begin{array}{llll} E_1^+\to b_1^+b_8^+ & E_2^+\to a_1^+a_2^+ & E_3^+\to a_2^+a_1^- & E_4^+\to a_3^+a_2^- \\ E_5^+\to a_4^+a_3^- & E_6^+\to a_5^+a_4^- & & \end{array} \tag{15a}$$

and

$$H_1\to M_1+M_8 \qquad H_2\to N_1+N_2 \qquad H_i\to N_{i-1}-N_{i-2} \qquad (3\leqslant i\leqslant 6). \tag{15b}$$

The Cartan matrix of F_4 is given by (see [6])

$$A=\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \tag{16}$$

$$E_1^+\to a_1^+a_2^- \qquad E_2^+\to a_2^+a_3^- \qquad E_3^+\to\sqrt{2}a_3^+ \qquad E_4^+\to\sqrt{2}b_4^+ \tag{17a}$$

and

$$H_1 \to N_1 - N_2 \qquad H_2 \to N_2 - N_3 \qquad H_3 \to 2N_3 \qquad H_4 \to 2M_4. \tag{17b}$$

It is easy to verify that all the relations (1) and (2) defining the universal enveloping algebra $U_q(F_4)$ associated to matrix (16) are satisfied.

Let us check them for the last row of the Cartan matrix since the remaining equations correspond to the Cartan matrix of B_3 (see [3]):

$a_{41} = 0$:

$$\sum_{n=0}^{1} (-1)^n \begin{bmatrix} 1 \\ n \end{bmatrix}_{q^2} (E_4^+)^{1-n} E_1^+ (E_4^+)^n = [E_4^+, E_1^+] \approx [b_4^+, a_1^+ a_2^-] = [b_4^+, b_1^+ b_2^-] = 0 \tag{18a}$$

$a_{42} = 0$:

$$\sum_{n=0}^{1} (-1)^n \begin{bmatrix} 1 \\ n \end{bmatrix}_{q^2} (E_4^+)^{1-n} E_2^+ (E_4^+)^n = [E_4^+, E_2^+] \approx [b_4^+, a_2^+ a_3^-] = [b_4^+, b_2^+ b_3^-] = 0 \tag{18b}$$

$a_{43} = -1$:

$$\sum_{n=0}^{1} (-1)^n \begin{bmatrix} 1 \\ n \end{bmatrix}_{q^2} (E_4^+)^{1-n} E_3^+ (E_4^+)^n$$

$$= [E_4^+, [E_4^+, E_3^+]_{q^{-2}}]_{q^2} \approx [b_4^+, [b_4^+, a_3^+]_{q^{-2}}]_{q^2} = 0. \tag{18c}$$

We hope that this work will be useful in the construction of the R-matrix for the exceptional algebras. Let us also mention, among the different developments of this work, the realization of the quantum affine algebras using an affinization of the q-deformed Clifford algebra.

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